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JORDAN IDEALS WITH MULTIPLICATIVE DERIVATIONS IN 3-PRIME NEAR-RINGS

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Abstract. In this work, we investigate the commutativity of 3-prime near-rings satisfying some differential identities on Jordan ideals involving multiplicative derivation. Some well-known results characterizing commutativity of 3-prime near-rings by derivations have been generalized by using multiplicative derivation. Further, we discuss an example to prove that the necessity of the 3-primeness hypothesis imposed on the various theorems cannot be marginalized.

Keywords: prime near-rings, multiplicative derivations, commutativity.

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1 Introduction

In this paper, $\mathcal{N}$ will represent a right near-ring. A near-ring $\mathcal{N}$ is called a zero-symmetric if $x.0 = 0$ for all $x \in \mathcal{N}$ (recall that a right distributivity yields $0.x = 0$). We note that for a right near-rings, $-(xy) = (-x)y$ for all $x, y \in \mathcal{N}$. Therefore, near rings are generalized rings, need not be commutative, and most importantly, only one distributive law is postulated (e.g., Example 1.4, Pilz (1983)). $\mathcal{N}$ is called a 3-prime if for all $x, y \in \mathcal{N}$, $x\mathcal{N}y = \{0\}$ implies $x = 0$ or $y = 0$. A near ring $\mathcal{N}$ is known as 2-torsion free if $2x = 0$ implies $x = 0$ holds for all $x \in \mathcal{N}$. The symbol $\mathcal{Z}(\mathcal{N})$ will represent the multiplicative centre of $\mathcal{N}$, that is, $\mathcal{Z}(\mathcal{N}) = \{x \in \mathcal{N}/xy = yx$ for all $y \in \mathcal{N}\}$. We will write for all $x, y \in \mathcal{N}, [x, y] = xy - yx$ and $x \circ y = xy + yx$ for the Lie products and Jordan products, respectively. In Boua et al. (2014) the notion of Jordan ideal defined as the following: An additive subgroup $\mathcal{J}$ of $\mathcal{N}$ is said to be a Jordan ideal of $\mathcal{N}$ if $j \circ n \in \mathcal{J}$ and $n \circ j \in \mathcal{J}$ for all $j \in \mathcal{J}, n \in \mathcal{N}$. A derivation d on a near-ring $\mathcal{N}$ is a group endomorphism on $(\mathcal{N}, +)$ which satisfies $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$, or equivalently, as noted in Wang (1994), that $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$. The notion of derivation is generalized in different directions by several authors (see, for example, Ashraf & Ali (2008); Bell (1997); Daif (1991); Martindale (1969); Oukhtite & Raji (2019); Boua et al. (2017); Raji (2021); Wang (1994) where further references can be found). In Daif (1991) the notion of multiplicative derivation of a ring $\mathcal{R}$ was introduced by Daif motivated by Martindale in Martindale (1969). A mapping $d : \mathcal{R} \to \mathcal{R}$, not necessarily additive, is called a multiplicative derivation if $d(xy) = xd(y)+d(x)y$ holds for all $x, y \in \mathcal{R}$. In Goldmann & Šemrl (1996), the authors studied these mappings and provided the full description of such mappings. Clearly, every derivation on a near-ring is a multiplicative derivation, but the converse statement does not hold in general. For more details, see for instance, Ashraf & Siddeeque (2019), examples 1.1, 1.2 and Bedir et al. (2017), where further references can be found.

Being motivated by this difference, we continue this line of investigation and we study the structure of 3-prime near-rings in which multiplicative derivations satisfy certain identities involving Jordan ideal.

2 Main results

We facilitate our discussion with the following lemmas which are required for developing the proofs of our main theorems. Note that the proofs of Lemmas 1, 2, 4 and 5 can be seen in (Boua et al., 2014, Lemma 2), (Boua et al., 2014, Lemma 1), (Kamal & Al-Shaalan, 2013, Lemma 2.1) and (Bedir et al., 2017, Lemma 3), respectively.

Lemma 1. *Let $\mathcal{N}$ be 2-torsion free 3-prime near-ring, and $\mathcal{J}$ a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{J} \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

Lemma 2. *Let $\mathcal{N}$ be 3-prime near-ring and $\mathcal{J}$ a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{J}x = \{0\}$, then $x = 0$.*

Lemma 3. *Let $\mathcal{N}$ be a 3-prime near-ring admitting a nonzero multiplicative derivation d, then $\mathcal{N}$ satisfies the following partial distribution law*

$$z(xd(y) + d(x)y) = zxd(y) + zd(x)y \text{ for all } x, y, z \in \mathcal{N}.$$

Proof. A proof can be using a similar approach as in (Bell & Mason, 1987, Lemma 1). □

Lemma 4. *A near-ring $\mathcal{N}$ admitting a multiplicative derivation d if and only if it is zero-symmetric.*

Lemma 5. *Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero multiplicative derivation d such that $ad(\mathcal{N}) = 0$ or $d(\mathcal{N})a = 0$, then $a = 0$.*

In Boua et al. (2014) proved that if a 2-torsion free 3-prime near-ring $\mathcal{N}$ admitting a nonzero derivation d satisfies one of the conditions: *i)* $[d(n), j] = 0$, *ii)* $d([n, j]) = 0$ for all $n \in \mathcal{N}, j \in \mathcal{J}$, where $\mathcal{J}$ is a nonzero Jordan ideal of $\mathcal{N}$, then $\mathcal{N}$ is a commutative ring. In this line, (Asma & Inzamam, 2020, Theorem 7) extended this study to a multiplicative derivation, precisely, they showed that a 2-torsion free 3-prime near-ring $\mathcal{N}$ must be a commutative ring or $d = 0$ on $\mathcal{J}$ if $\mathcal{N}$ admits a multiplicative derivation d satisfying one of the conditions above. However, this results are less precise. The following Theorems treats the above conditions and give the corrected results.

Theorem 1. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and $\mathcal{J}$ be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero multiplicative derivation d such that $[d(x), j] = 0$ for all $x \in \mathcal{N}, j \in \mathcal{J}$. Then at least one of the following assertions holds:*
i) $(\mathcal{J}, +)$ is a commutative group and $\mathcal{J}$ contains a nonzero commutative ring.
ii) $\mathcal{N}$ is a commutative ring.

Proof. Assume that

$$d(x)j = jd(x) \text{ for all } x \in \mathcal{N}, j \in \mathcal{J}.$$

Replacing x by xi, where $i \in \mathcal{J}$, and using Lemma 3, we get

$$xd(i)j + d(x)ij = jxd(i) + jd(x)i \text{ for all } x \in \mathcal{N}, i, j \in \mathcal{J},$$

it follows that

$$xd(i)j + d(x)ij = jxd(i) + d(x)ji \text{ for all } x \in \mathcal{N}, i, j \in \mathcal{J}. \tag{1}$$

Taking $i = j$ in (1), we get

$$xjd(j) = jxd(j) \text{ for all } x \in \mathcal{N}, j \in \mathcal{J}. \tag{2}$$

Substituting xt for x, where $t \in \mathcal{N}$, in (2), we obtain

$$xtjd(j) = jxtd(j) \text{ for all } t, x \in \mathcal{N}, j \in \mathcal{J},$$

which can be rewritten as,

$$[x, j]\mathcal{N}d(j) = \{0\} \text{ for all } x \in \mathcal{N}, j \in \mathcal{J}.$$

In view of 3-primeness of $\mathcal{N}$, the latter relation gives that

$$d(j) = 0 \text{ or } j \in \mathcal{Z}(\mathcal{N}) \text{ for all } j \in \mathcal{J}. \tag{3}$$

a) Suppose that $d(\mathcal{J}) = \{0\}$, in this case (1) will be

$$d(x)ij = d(x)ji \text{ for all } x \in \mathcal{N}, i, j \in \mathcal{J},$$

which implies that

$$ijd(x) = jid(x) \text{ for all } x \in \mathcal{N}, i, j \in \mathcal{J}.$$

And therefore,

$$[i, j]d(x) = 0 \text{ for all } x \in \mathcal{N}, i, j \in \mathcal{J}.$$

Using the fact that $d \neq 0$ together Lemma 5, we find that $ij = ji$ for all $i, j \in \mathcal{J}$, (i.e $\mathcal{J}$ commute under the multiplication of $\mathcal{N}$). Replacing i and j by $i+i$ and $j+p$, respectively, in the last result, we obtain $(i + i)(j + p) = (j + p)(i + i)$ for all $i, j, p \in \mathcal{J}$. By a simple calculation and after simplifying, we arrive at

$$\mathcal{J}((p + j) - (j + p)) = \{0\} \text{ for all } j, p \in \mathcal{J},$$

and therefore Lemma 3 shows that

$$p + j = j + p \text{ for all } j, p \in \mathcal{J}.$$

Consequently, $(\mathcal{J}, +)$ is a commutative group. Now, our goal is to show that $\mathcal{J}$ contains a commutative ring. Let $\mathcal{H}$ the subset of $\mathcal{J}$ be defined by:

$$\mathcal{H} = \{i \in \mathcal{J} \,/\, ij \in \mathcal{J} \text{ for all } j \in \mathcal{J}\}$$

Proves that $\mathcal{H} \neq \{0\}$. Indeed, For all $i, j \in \mathcal{J}$, we have $i \circ j = ij + ji = ji + ji = (j + j)i \in \mathcal{J}$ and therefore $(j + j) \in \mathcal{H}$ for all $j \in \mathcal{J}$. In view of 2-torsion freeness, we have $j_0 + j_0 \neq 0$ for all $j_0 \in \mathcal{J} \setminus \{0\}$ which implies that $\mathcal{H} \neq \{0\}$. Now, with a simple calculation, it is easy to check that $\mathcal{H}$ under the operations of $\mathcal{N}$ is a commutative ring.

b) If $d(\mathcal{J}) \neq 0$. In this case, there exists un element $i_0 \in \mathcal{J}$ such that $d(i_0) \neq 0$; then (3) implies that $i_0 \in \mathcal{Z}(\mathcal{N})$. Now, suppose there exists an element $j_0 \in \mathcal{J}$ such that $d(j_0) = 0$. Our goal is to prove that $j_0 \in \mathcal{Z}(\mathcal{N})$. Indeed, taking $x = xi_0$ and $j = j_0$ in (1), where $x \in \mathcal{N}$, we get

$$d(xi_0)j_0 = j_0 d(xi_0) \text{ for all } x \in \mathcal{N}.$$

Which implies that

$$xd(i_0)j_0 + d(x)i_0j_0 = j_0xd(i_0) + j_0d(x)i_0 \text{ for all } x \in \mathcal{N}.$$

After simplifying, we obtain

$$xj_0d(i_0) = j_0xd(i_0) \text{ for all } x \in \mathcal{N}. \tag{4}$$

Replacing x by xt, where $t \in \mathcal{N}$, in (4), we obtain

$$xj_0td(i_0) = j_0xtd(i_0) \text{ for all } x, t \in \mathcal{N},$$

this shows that

$$[x, j_0]\mathcal{N}d(i_0) = 0 \text{ for all } x \in \mathcal{N}. \tag{5}$$

By 3-primeness of $\mathcal{N}$ and the fact that $d(i_0) \neq 0$, (5) assure that $j_0 \in \mathcal{Z}(\mathcal{N})$. Accordingly, (3) reduces to $\mathcal{J} \subseteq \mathcal{Z}(\mathcal{N})$ and lemma 2 shows that $\mathcal{N}$ is a commutative ring, which completes the proof.

□

Remark 1. *Under the assumptions of the previous theorem, we deduce that $\mathcal{N}$ contains a nonzero commutative ring.*

Corollary 1. (Boua et al., 2014, Theorem 2) Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and $\mathcal{J}$ a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero derivation d such that $[d(n), j] = 0$ for all $n \in \mathcal{N}, j \in \mathcal{J}$, then $\mathcal{N}$ is a commutative ring.

In the following Theorems, we suppose the assumption $d(\mathcal{J}) \neq \{0\}$ so that the results of the theorems to make sense. Indeed, if the quoted condition is not true, then $\mathcal{N}$ equipped with a nonzero multiplicative derivation cannot be a commutative ring. To be convinced of this, it suffices to see that the condition $d(\mathcal{J}) = \{0\}$ implies $d(\mathcal{Z}(\mathcal{N})) = \{0\}$.

Theorem 2. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and $\mathcal{J}$ be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero multiplicative derivation d such that $d(\mathcal{J}) \neq \{0\}$ and $d([x, j]) = 0$ for all $x \in \mathcal{N}, j \in \mathcal{J}$, then $\mathcal{N}$ is a commutative ring.*

Proof. By hypothesis, we have

$$d([x, j]) = 0 \text{ for all } x \in \mathcal{N}, j \in \mathcal{J}. \tag{6}$$

Taking xj instead of x in (6) and noting that $[xj, j] = [x, j]j$, we get

$$d([x, j]j) = [x, j]d(j) = 0 \text{ for all } x \in \mathcal{N}, j \in \mathcal{J},$$

so that

$$xjd(j) = jxd(j) \text{ for all } x \in \mathcal{N}, j \in \mathcal{J}. \tag{7}$$

Replacing x by xt in (7), where $t \in \mathcal{N}$, and using it again, we obtain

$$xjtd(j) = jxtd(j) \text{ for all } x, t \in \mathcal{N}, j \in \mathcal{J}.$$

It follows that

$$[x, j]\mathcal{N}d(j) = \{0\} \text{ for all } x \in \mathcal{N}, j \in \mathcal{J}. \tag{8}$$

In the light of the 3-primeness of $\mathcal{N}$, (8) forces

$$d(j) = 0 \text{ or } j \in \mathcal{Z}(\mathcal{N}) \text{ for all } j \in \mathcal{J}. \tag{9}$$

By hypothesis, we have $d(\mathcal{J}) \neq 0$, so there exist $j_0 \in \mathcal{J}$ such that $d(j_0) \neq 0$, then from (9) it follows that $j_0 \in \mathcal{Z}(\mathcal{N})$. Substituting j_0x for x in (6), we get

$$xjd(j_0) = jxd(j_0) \text{ for all } x \in \mathcal{N}, j \in \mathcal{J}. \tag{10}$$

Replacing x by xt in (10), where $t \in \mathcal{N}$, we obtain

$$[x, j]\mathcal{N}d(j_0) = 0 \text{ for all } x \in \mathcal{N}, j \in \mathcal{J}.$$

Since $\mathcal{N}$ is 3-prime, we get

$$d(j_0) = 0 \text{ or } j \in \mathcal{Z}(\mathcal{N}) \text{ for all } j \in \mathcal{J}. \tag{11}$$

As $d(j_0) \neq 0$, (11) forces $j \in \mathcal{J}$ for all $j \in \mathcal{J}$, then $\mathcal{J} \subseteq \mathcal{Z}(\mathcal{N})$ and Lemma (1) shows that $\mathcal{N}$ is a commutative ring. This ends the proof. □

As an application of the above Theorem, we get the following corollary.

Corollary 2. (Ashraf & Ali, 2008, Theorem 4.1) Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and d a nonzero derivation. If $d([\mathcal{N},\mathcal{N}]) = \{0\}$, then $\mathcal{N}$ is a commutative ring.

Theorem 3. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and $\mathcal{J}$ be a nonzero Jordan ideal of $\mathcal{N}$. There is no nonzero multiplicative derivation d such that $d(\mathcal{J}) \neq \{0\}$ and $d(x \circ j) = 0$ for all $x \in \mathcal{N}, j \in \mathcal{J}$.*

Proof. Assume that $\mathcal{N}$ admits a nonzero multiplicative derivation d such that $d(\mathcal{J}) \neq \{0\}$ and

$$d(x \circ j) = 0 \text{ for all } x \in \mathcal{N}, j \in \mathcal{J}. \tag{12}$$

Replacing x by xj in(12), and noting that $xj \circ j = (x \circ j)j$, we get

$$(x \circ j)d(j) = 0 \text{ for all } x \in \mathcal{N}, j \in \mathcal{J},$$

after solving this expression, we find that

$$xjd(j) = -jxd(j) \text{ for all } x \in \mathcal{N}, j \in \mathcal{J}. \tag{13}$$

Substituting xt for x, where $t \in \mathcal{N}$, in (13) and using (13), we obtain

$$x(-j)td(j) = (-j)xtd(j) \text{ for all } x, t \in \mathcal{N}, j \in \mathcal{J}$$

and therefore

$$[x, -j]\mathcal{N}d(j) = \{0\} \text{ for all } x \in \mathcal{N}, j \in \mathcal{J}. \tag{14}$$

Since $\mathcal{N}$ is 3-prime near-ring, (14) implies that

$$d(j) = 0 \text{ or } -j \in \mathcal{Z}(\mathcal{N}) \text{ for all } j \in \mathcal{J}. \tag{15}$$

Suppose there is an element $j_0 \in \mathcal{J}$ such that $d(j_0) = 0$. As $d(\mathcal{J}) \neq 0$, there exists an element $i_0 \in \mathcal{J}$ such that $d(i_0) \neq 0$, then (15) assures that $-i_0 \in \mathcal{Z}(\mathcal{N})$. We prove that $d(-i_0) \neq 0$. In fact, suppose that $d(-i_0) = 0$ and putting $-i_0$ instead of j in (12), we get

$$d(x \circ (-i_0)) = 0 \text{ for all } x \in \mathcal{N}$$

that is, we have that

$$d((x + x)(-i_0)) = d(x + x)(-i_0) = 0 \text{ for all } x \in \mathcal{N}.$$

Right-multiplying by t, where $t \in \mathcal{N}$, we obtain

$$d((x + x)t(-i_0)) = 0 \text{ for all } x, t \in \mathcal{N}.$$

Which reduces to

$$d(x + x)\mathcal{N}(-i_0) = \{0\} \text{ for all } x \in \mathcal{N}.$$

In view of 3-primeness of $\mathcal{N}$, the latter equation shows that

$$d(x + x) = 0 \text{ or } -i_0 = 0 \text{ for all } x \in \mathcal{N}. \tag{16}$$

It is clear that the second case of (16) cannot be required. Indeed, suppose that $-i_0 = 0$; then we will have $i_0 = 0$ which implies that $d(i_0) = 0$, a contradiction. Now, we treat the first case of (16) that is, we have that $d(x + x) = 0$ for all $x \in \mathcal{N}$. Writing $x(-i_0)$ instead of x in previous expression and using it again, we arrive at

$$(2(-i_0))d(x) = 0 \text{ for all } x \in \mathcal{N}.$$

Since $d \neq 0$, the last equation together with Lemma 5 and 2-torsion freeness of $\mathcal{N}$ forces $-i_0 = 0$, a contradiction again. Consequently, $d(-i_0) \neq 0$. On the other hand, by hypothesis, we have $d(x \circ j_0) = 0$ for all $x \in \mathcal{N}$. Putting $x(-i_0)$ instead of x in the last equation, we arrive at

$$xj_0d(-i_0) = -j_0xd(-i_0) \text{ for all } x \in \mathcal{N}. \tag{17}$$

Substituting xt for x, where $t \in \mathcal{N}$, in (17) and using (17), we arrive at

$$[x, -j_0]\mathcal{N}d(-i_0) = \{0\} \text{ for all } x \in \mathcal{N}. \tag{18}$$

Using the 3-primeness of $\mathcal{N}$, we find that

$$d(-i_0) = 0 \text{ or } -j_0 \in \mathcal{Z}(\mathcal{N}). \tag{19}$$

Since $d(-i_0) \neq 0$, equation (21) yields $-j_0 \in \mathcal{Z}(\mathcal{N})$. Accordingly, (15) reduces to $-j \in \mathcal{Z}(\mathcal{N})$ for all $j \in \mathcal{J}$. Replacing j by $-j$ and applying Lemma 2, we conclude that $\mathcal{N}$ is a commutative ring and therefore our hypothesis reduces to $d(xj + xj) = d((x + x).j) = (x + x)d(j) + d(x + x)j = 0$ for all $x \in \mathcal{N}, j \in \mathcal{J}$. Substituting xj for x in the last equation and by defining property of d, we get $(j + j)xd(j) = 0$ for all $x \in \mathcal{N}, j \in \mathcal{J}$ which, as $\mathcal{N}$ is 2-torsion free, implies that $jxd(j) = 0$ for all $j \in \mathcal{J}$, which can be rewritten as $j\mathcal{N}d(j) = 0$ for all $j \in \mathcal{J}$. Once again, using the 3-primeness, we conclude that $j = 0$ or $d(j) = 0$ for all $j \in \mathcal{J}$. Hence, in both cases, we have $d(j) = 0$ for all $j \in \mathcal{J}$ which is contrary to our hypothesis. This complete the proof of our Theorem. □

Theorem 4. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and $\mathcal{J}$ be a nonzero Jordan ideal of $\mathcal{N}$, then $\mathcal{N}$ admits no nonzero multiplicative derivation d such that $d(\mathcal{J}) \neq 0$ and $d(x \circ j) = x \circ j$ for all $x \in \mathcal{N}, j \in \mathcal{J}$.*

Proof. We are given that $d(\mathcal{J}) \neq \{0\}$ and

$$d(x \circ j) = x \circ j \text{ for all } x \in \mathcal{N}, j \in \mathcal{J}. \tag{20}$$

Substituting xj for x in (20), and after simplifying we get

$$(x \circ j)d(j) = 0 \text{ for all } x \in \mathcal{N}, j \in \mathcal{J},$$

that is

$$xjd(j) = -jxd(j) \text{ for all } x \in \mathcal{N}, j \in \mathcal{J}.$$

Writing $x = xt$, where $t \in \mathcal{N}$, and using again the latter equation, we obtain

$$x(-j)td(j) = (-j)xtd(j) \text{ for all } x, t \in \mathcal{N}, j \in \mathcal{J}$$

it follows that

$$[x, -j]\mathcal{N}d(j) = \{0\} \text{ for all } x \in \mathcal{N}, j \in \mathcal{J}. \tag{21}$$

In view of the 3-primeness of $\mathcal{N}$, (21) shows that

$$d(j) = 0 \text{ or } -j \in \mathcal{Z}(\mathcal{N}) \text{ for all } j \in \mathcal{J}. \tag{22}$$

Suppose there is $j_0 \in \mathcal{J}$ such that $d(j_0) = 0$. As $d(\mathcal{J}) \neq \{0\}$, there is an element $i_0 \in \mathcal{J}$ such that $d(i_0) \neq 0$ and therefore $-i_0 \in \mathcal{Z}(\mathcal{N})$ by (22) . Now, we prove that $d(-i_0) \neq 0$. Indeed, suppose that $d(-i_0) = 0$, in this case, replacing j by $-i_0$ in (20), we arrive at

$$d(x + x)(-i_0) = (x + x)(-i_0) \text{ for all } x \in \mathcal{N},$$

Now, right multiplying the previous relation by n, where $n \in \mathcal{N}$, and using the fact $-i_0 \in \mathcal{Z}(\mathcal{N})$, we find that $[d(x+x),\,(x+x)]n(-i_0) = 0$ for all $x, n \in \mathcal{N}$ which implies that

$$[d(x+x),\,(x+x)]\mathcal{N}(-i_0) = \{0\} \text{ for all } x \in \mathcal{N}.$$

In view of the 3-primeness of $\mathcal{N}$ and $-i_0 \neq 0$, the last relation shows that

$$d(x+x) = (x+x) \text{ for all } x \in \mathcal{N}. \tag{23}$$

Replacing x by $x(-i_0)$ in (23) and noting that $x(-i_0) + x(-i_0) = \big((-i_0) + (-i_0)\big)x$ we get

$$\big((-i_0)+(-i_0)\big)d(x)+d((-i_0)+(-i_0))x = \big((-i_0)+(-i_0)\big)d(x)+\big((-i_0)+(-i_0)\big)x = \big((-i_0)+(-i_0)\big)x.$$

It follows that $\big((-i_0) + (-i_0)\big)d(\mathcal{N}) = \{0\}$. Invoking Lemma 5 and using the fact that $d \neq 0$, we obtain $\big((-i_0) + (-i_0)\big) = 0$ which, because $\mathcal{N}$ is 2-torsion free, implies that $(-i_0) = 0$, a contradiction and therefore $d(-i_0) \neq 0$. Now, returning to (20) and replacing respectively x and j by $x(-i_0)$ and j_0 and using (20), we get $d\big(x(-i_0) \circ j_0\big) = x(-i_0) \circ j_0$. Since $(-i_0) \in \mathcal{Z}(\mathcal{N})$, it follows that

$$\begin{aligned} d\big(x(-i_0) \circ j_0\big) &= d\big((x \circ j_0)(-i_0)\big) \\ &= (x \circ j_0)d(-i_0) + d(x \circ j_0)(-i_0) \\ &= (x \circ j_0)d(-i_0) + (x \circ j_0)(-i_0) \\ &= (x \circ j_0)(-i_0) \end{aligned}$$

Consequently, we conclude that

$$(x \circ j_0)d(-i_0) = 0 \text{ for all } x \in \mathcal{N}. \tag{24}$$

It is clair that equation (24) is similar to (17) in the previous theorem, and therefore using the same approach as used previously we arrive at $-j_0 \in \mathcal{Z}(\mathcal{N})$, then (22) reduces to $\mathcal{J} \subseteq \mathcal{Z}(\mathcal{N})$. Accordingly, $\mathcal{N}$ is a commutative ring by Lemma 1. Hence, equation (20) yields

$$\begin{aligned} d(x \circ j) &= d((x+x)j) \\ &= (x+x)d(j) + d(x+x)j \\ &= (x+x)j \quad \text{for all} \quad j \in \mathcal{J}, x \in \mathcal{N}. \end{aligned}$$

Taking $x = xj$ in the previous relations and using (20) together with 2-torsion freeness, we find that $jxd(j) = 0$ for all $j \in \mathcal{J}, x \in \mathcal{N}$ which implies that $j\mathcal{N}d(j) = \{0\}$ for all $j \in \mathcal{J}$. Since $\mathcal{N}$ is 3-prime, this implies that $j = 0$ or $d(j) = 0$. In the both cases we have $d(j) = 0$ and whence it follows that $d(\mathcal{J}) = \{0\}$ which contradicts our original assumption that $d(\mathcal{J}) \neq \{0\}$. This ends the proof. □

If we consider the case where $\mathcal{J} = \mathcal{N}$, we obtain a result on derivations as a special case as follows.

Corollary 3. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. There is no derivation d of $\mathcal{N}$ such that $d(x \circ y) = x \circ y$ for all $x, y \in \mathcal{N}$.*

The following example demonstrates that the 3-primeness assumption is essential in the hypotheses of the our theorems.

Example 1. *Let $\mathcal{S}$ be a noncommutative nonzero unit ring. Let us define $\mathcal{N}$, $\mathcal{J}$, $d : \mathcal{N} \to \mathcal{N}$ by:*

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid 0, x, y \in \mathcal{S} \right\},$$

$$\mathcal{J} = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & k \\ 0 & 0 & 0 \end{pmatrix} \mid 0, x, k \in \mathcal{S} \textit{ with } k \in \{0, 1, -1\} \right\},$$

and

$$d \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & xy \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to verify that $\mathcal{N}$ is a non 3-prime near-ring, $\mathcal{J}$ is a nonzero Jordan ideal of $\mathcal{N}$ and $d(\mathcal{J}) \neq \{0\}$. Moreover, d is a nonzero multiplicative derivation which satisfies the properties:

1. $[d(A), J] = 0$ *for all* $A \in \mathcal{N}, J \in \mathcal{J}$,
2. $d([A, J]) = 0$ *for all* $A \in \mathcal{N}, J \in \mathcal{J}$,
3. $d(A \circ J) = 0$ *for all* $A \in \mathcal{N}, J \in \mathcal{J}$,
4. $d(A \circ J) = A \circ J$ *for all* $A \in \mathcal{N}, J \in \mathcal{J}$.

But, since the multiplicative law in $\mathcal{N}$ is not commutative, then $\mathcal{N}$ cannot be a commutative ring.

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